The Central Limit Theorem

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Kerrich's experiment

- A South African mathematician named John Kerrich was visiting Copenhagen in 1940 when Germany invaded Denmark
- Kerrich spent the next five years in an internment camp
- To pass the time, he carried out a series of experiments in probability theory
- One of them involved flipping a coin 10,000 times

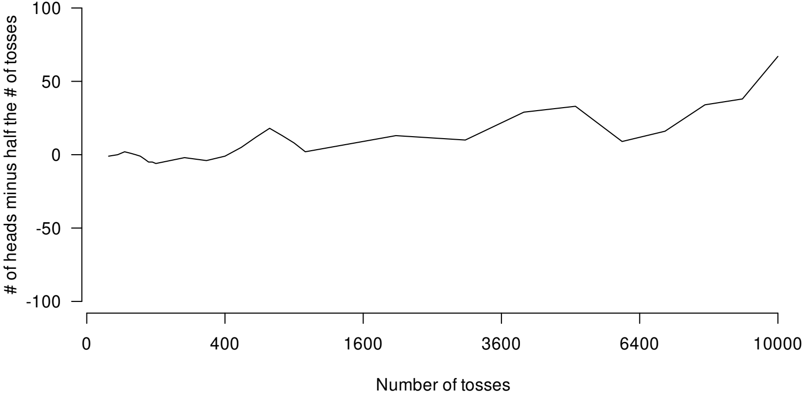
The law of averages

- We know that a coin lands heads with probability 50%
- Thus, after many tosses, the law of averages says that the number of heads should be about the same as the number of tails . . .
- . . . or does it?

Kerrich's results

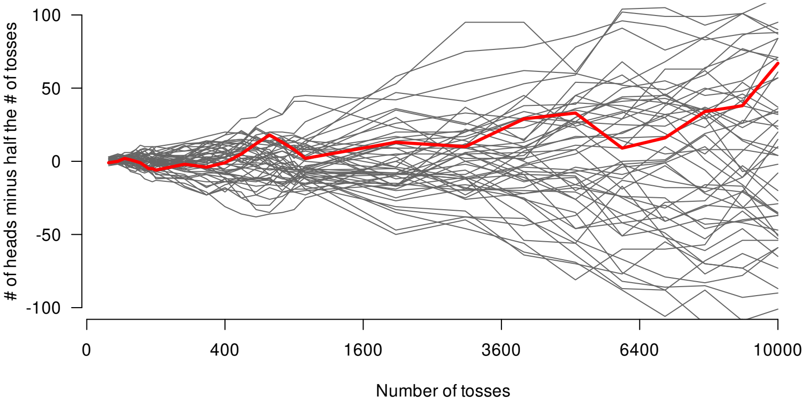
Number of tosses (n)	Number of heads	Heads - $0.5 \cdot$Tosses
10	4	-1
100	44	-6
500	255	5
1,000	502	2
2,000	1,013	13
3,000	1,510	10
4,000	2,029	29
5,000	2,533	33
6,000	3,009	9
7,000	3,516	16
8,000	4,034	34
10,000	5,067	67

Kerrich's results plotted



Instead of getting closer, the numbers of heads and tails are getting farther apart

Repeating the experiment 50 times

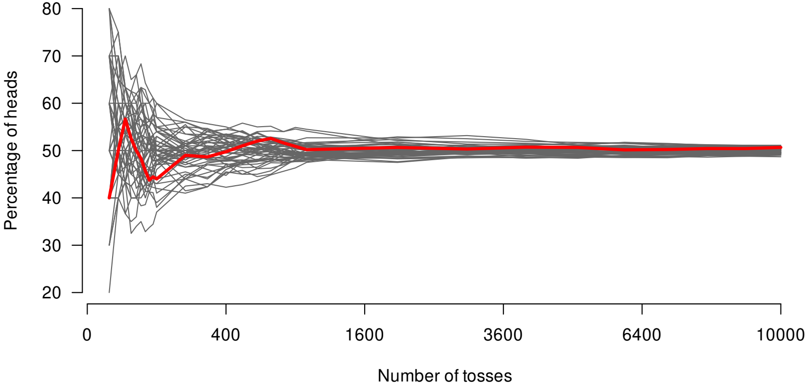


This is not a fluke — instead, it occurs systematically and consistently in repeated simulated experiments

Where's the law of averages?

- So where's the law of averages?
- Well, the law of averages does **not** say that as n increases the number of heads will be close to the number of tails
- What it says instead is that, as n increases, the average number of heads (i.e., the percentage of heads) will get closer and closer to the long-run average (in this case, 0.5 or 50%)
- The technical term for this is that the sample average, which is an estimate, *converges* to the "expected value" (more on this later), a parameter

Repeating the experiment 50 times, Part II



Trends in Kerrich's experiment

- There are three very important trends to keep track of in this experiment
- For any given n (number of tosses), consider the distribution of $\hat{\pi}$ values (percentage of heads; the gray lines in the figures)
    - What is the center of that distribution?
    - What is the spread?
    - What is the shape?

The expected value

- The center of a distribution is known as its *expected value*
- This is essentially the same thing as the mean, except instead of an average of observed values, the average is over all the possible values of the distribution
- In the coin-tossing experiment, we can see that this center never changes — it's always 50%, regardless of n

Individual versus average

- In fact, even for a single toss, the expected value is 0.5 (equal probability of 1 head and zero heads)
- To summarize, then: the expected value of the average is the same as the expected value for an individual, regardless of the number of tosses included in the average

The variability of the average

- This is not the case for the variability, however
- As we can see from the earlier figures, as n goes up, the variability of the *total* goes up, but the variability of the *average* goes down
- Indeed, the variability goes to 0 as n gets larger and larger — this is the law of averages
- The standard deviation of the average is given a special name in statistics to distinguish it from the sample standard deviation of data
- The SD of the mean is called the *standard error*
- The term *standard error* refers to the variability of any estimate, to distinguish it from the variability of individual tosses or people

The square root law

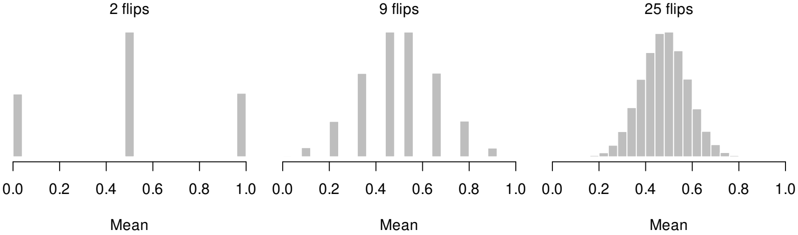
- The relationship between the variability of an individual (toss) and the variability of the average (of a large number of tosses) is a very important relationship, sometimes called the *square root law*:

$$\text{SE} = \frac{\text{SD}}{\sqrt{n}},$$

  where SE is the standard error of the mean and SD is the standard deviation of an individual (toss)
- This relationship holds for all averages, not just tosses of a coin

The distribution of the mean

Finally, let's look at the shape of this distribution by creating histograms of the mean in our simulation



The central limit theorem

- In summary, there are three very important phenomena going on here concerning the distribution of the sample average:
  (1) Its center is always equal to the expected value for an individual
  (2) Its variability (standard error) is always equal to the standard deviation divided by the square root of n
  (3) As n gets larger, the sampling distribution looks more and more like the normal distribution
- Furthermore, these three properties of the sampling distribution of the sample average hold for ***any distribution*** — not just the binomial

The central limit theorem (cont'd)

- This result is called the *central limit theorem*, and it is one of the most important, remarkable, and powerful results in all of statistics
- In the real world, we rarely know the distribution of our data
- But the central limit theorem says: we don't have to

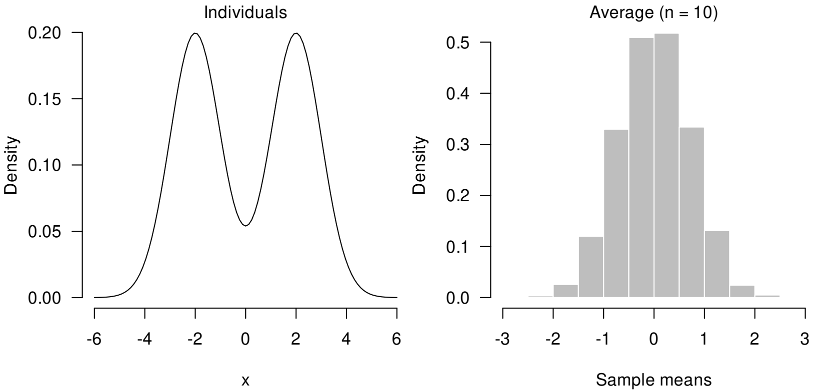
The central limit theorem (cont'd)

- Furthermore, as we have seen, knowing the mean and standard deviation of a distribution that is approximately normal allows us to calculate anything we wish to know with tremendous accuracy — and the sampling distribution of the mean is always approximately normal
- The only caveats:
  - Observations must be independent
  - The central limit theorem applies to the distribution of the mean — not necessarily to the distribution of other statistics
  - How large does n have to be before the distribution becomes close enough in shape to the normal distribution?

How large does n have to be?

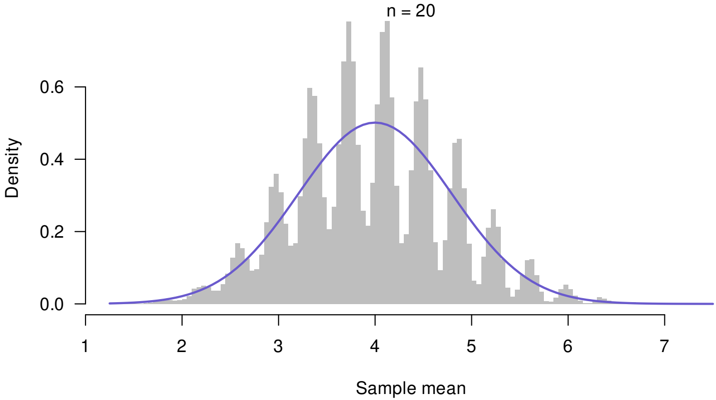
- Rules of thumb are frequently recommended that $n = 20$ or $n = 30$ is "large enough" to be sure that the central limit theorem is working
- There is some truth to such rules, but in reality, whether n is large enough for the central limit theorem to provide an accurate approximation to the true sampling distribution depends on how close to normal the original distribution is
- If the original distribution is close to normal, $n = 2$ might be enough
- If the underlying distribution is highly skewed or contains outliers, $n = 50$ might not be enough

Example #1

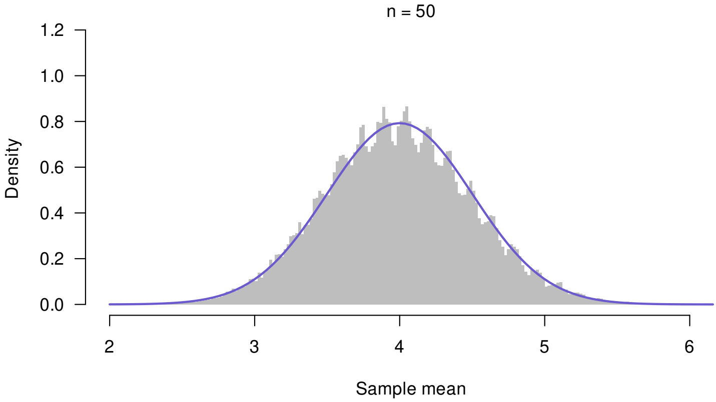

Individuals
Density
0.20
0.15
0.10
0.05
0.00
-6
-4
-2
0
2
4
6
x
Average (n = 10)
Density
0.5
0.4
0.3
0.2
0.1
0.0
-3
-2
-1
0
1
2
3
Sample means

Example #2

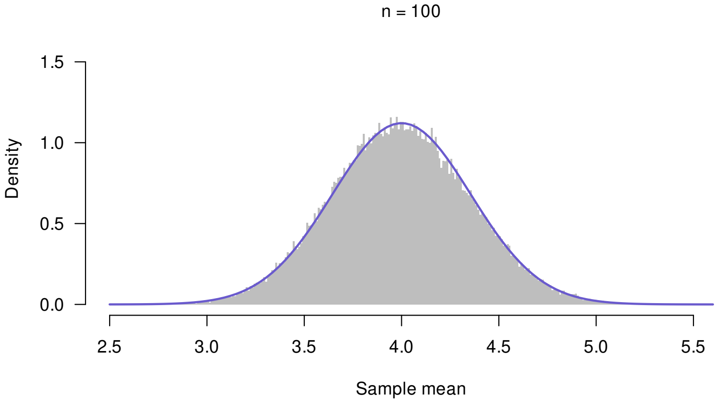
Now imagine an urn containing the numbers 1, 2, and 9:



Example #2 (cont'd)



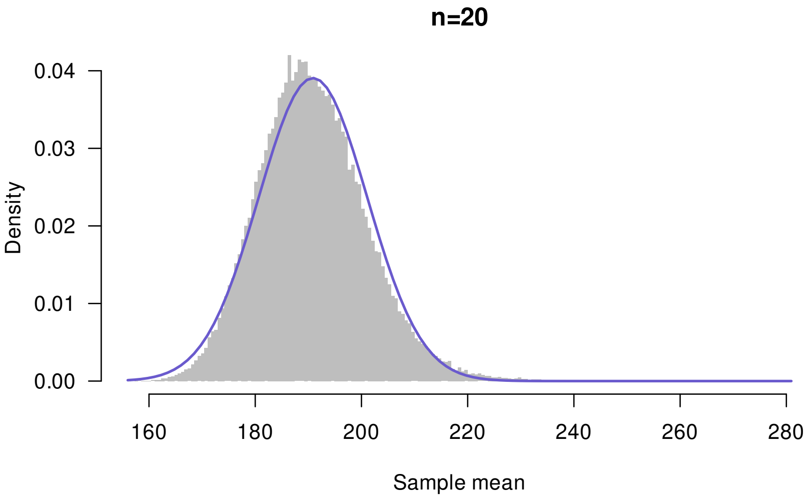
Example #2 (cont'd)



Example #3

- Weight tends to be skewed to the right (more people are overweight than underweight)
- Let's perform an experiment in which the NHANES sample of adult men is the population
- I am going to randomly draw twenty-person samples from this population (*i.e.* I am re-sampling the original sample)

Example #3 (cont'd)



Why do so many things follow normal distributions?

- We can see now why the normal distribution comes up so often in the real world: any time a phenomenon has many contributing factors, and what we see is the average effect of all those factors, the quantity will follow a normal distribution
- For example, there is no one cause of height — thousands of genetic and environmental factors make small contributions to a person's adult height, and as a result, height is normally distributed
- On the other hand, things like eye color, cystic fibrosis, broken bones, and polio have a small number of (or a single) contributing factors, and do not follow a normal distribution

Summary

- Central limit theorem:
  - The expected value of the average is always equal to the expected value of an individual
  - $\mathrm{SE} = \mathrm{SD}/\sqrt{n}$
  - As n gets larger, the sampling distribution looks more and more like the normal distribution
- Generally speaking, the sampling distribution looks pretty normal by about $n = 20$, but this could happen faster or slower depending on the population and how skewed it is