

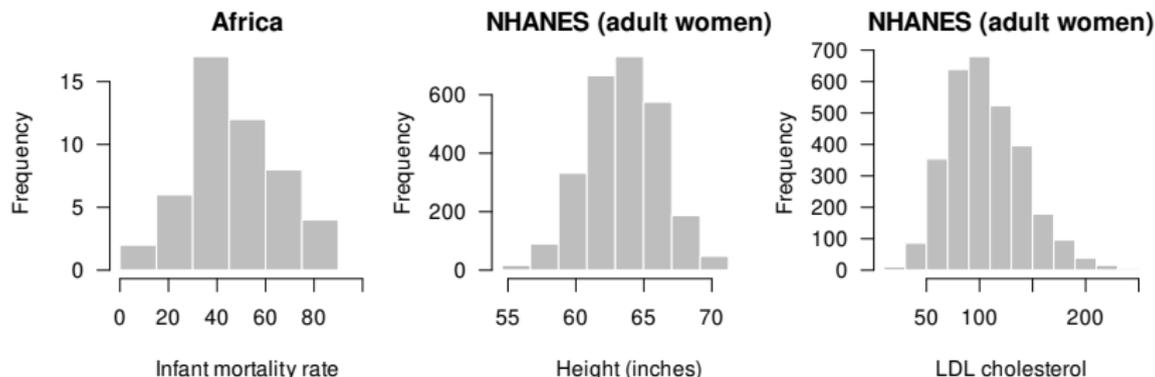
# The normal distribution

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# A common histogram shape

Histograms of infant mortality rates, heights, and cholesterol levels:



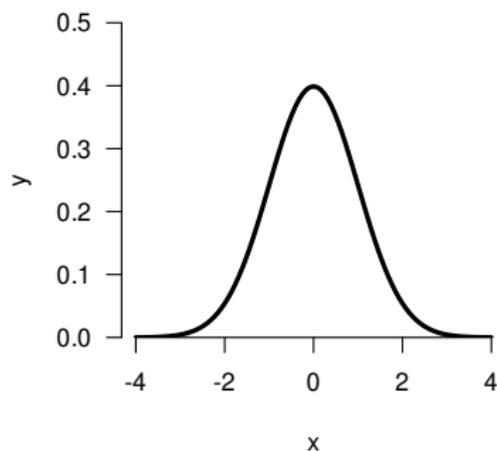
What do these histograms have in common?

# The normal curve

Mathematicians discovered long ago that the equation

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

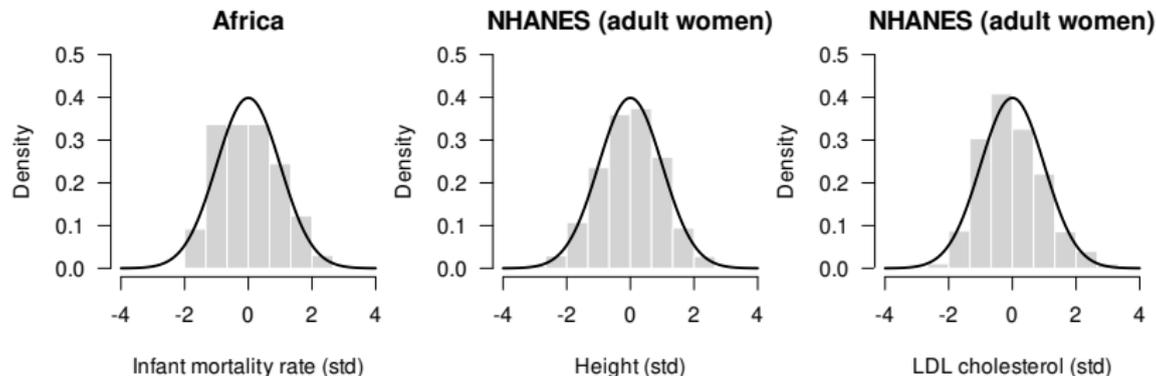
described the histograms of many random variables



# Features of the normal curve

- The normal curve is symmetric around  $x = 0$
- The normal curve drops rapidly down near zero as  $x$  moves away from 0
- The normal curve is always positive

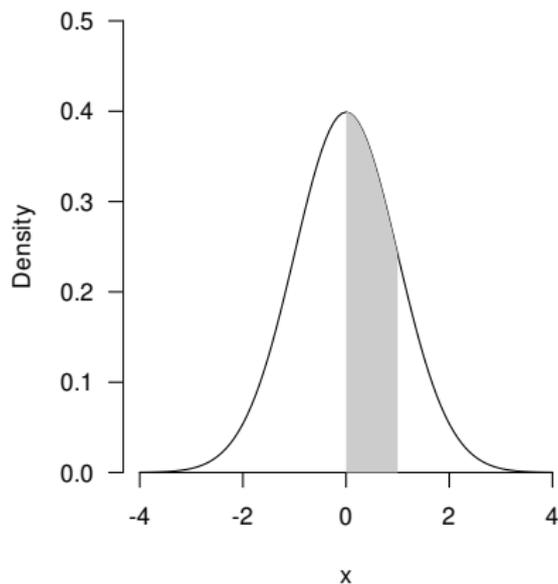
# The normal curve in action



- Technical note: The data has been standardized and the vertical axis is now "density"
- Data whose histogram looks like the normal curve are said to be *normally distributed* or to *follow a normal distribution*

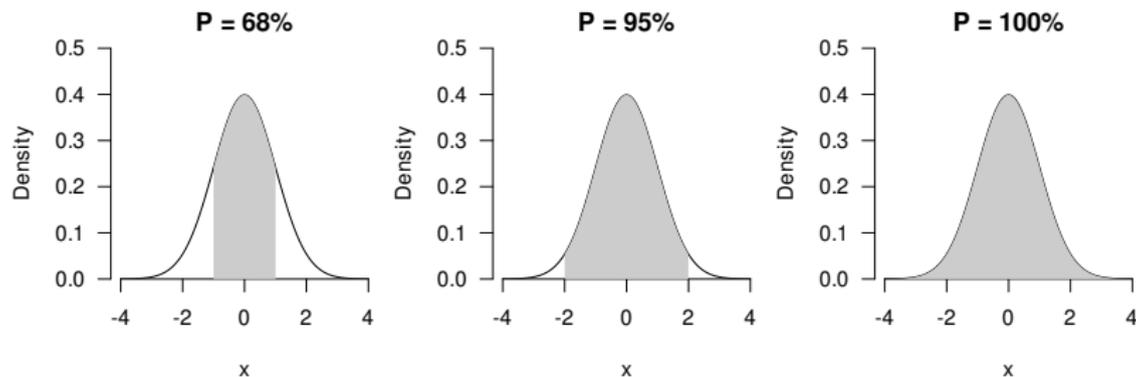
# Probabilities from the normal curve

Probabilities are given by the area under the normal curve:



# The 68%/95% rule

This is where the 68%/95% rule of thumb that we discussed earlier comes from:

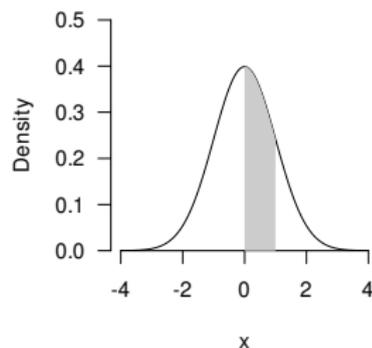
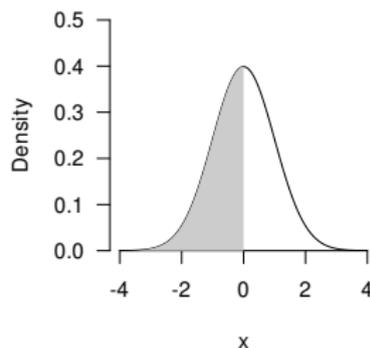
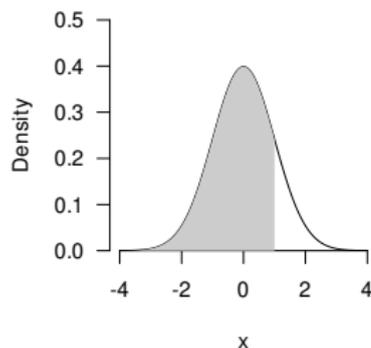


# Calculating probabilities

- By knowing that the total area under the normal curve is 1, we can get a rough idea of the area under a curve by looking at a plot
- However, to get exact numbers, we will need a computer
- “How much area is under this normal curve?” is an extremely common question in statistics, and programmers have developed algorithms to answer this question very quickly
- The output from these algorithms is commonly collected into tables, which is what you will have to use for exams

# Calculating the area under a normal curve, example 1

Find the area under the normal curve between 0 and 1

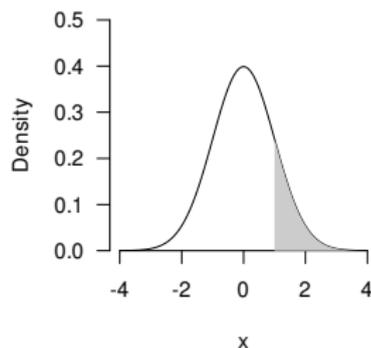
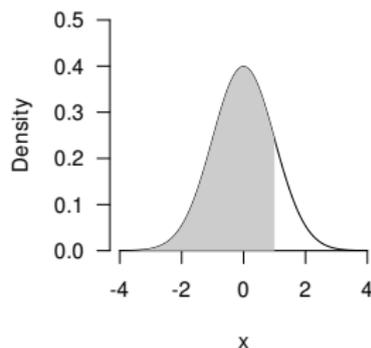
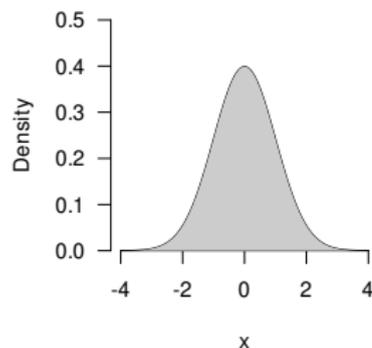


$$.84 - .5 = .34$$

```
pnorm(1) - pnorm(0)
# [1] 0.3413447
```

# Calculating the area under a normal curve, example 2

Find the area under the normal curve above 1

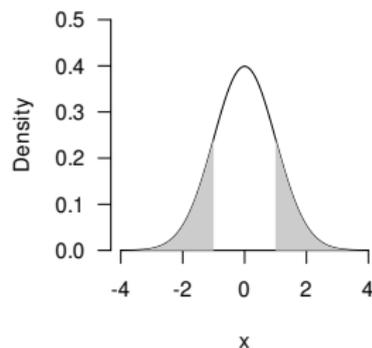
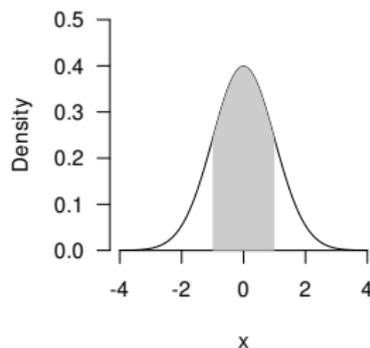
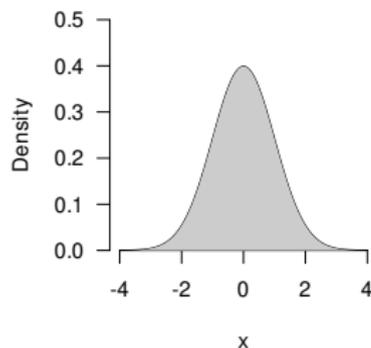


$$1 - 0.84 = 0.16$$

```
1 - pnorm(1)
# [1] 0.1586553
```

# Calculating the area under a normal curve, example 3

Find the area under the normal curve that lies outside -1 and 1



- $1 - (.84 - .16) = .32$
- Alternatively, we could have used symmetry:  $2(.16) = .32$

```
2 * pnorm(-1)
# [1] 0.3173105
```

# Calculating percentiles

- A related question of interest is, “What is the  $x$ th percentile of the normal curve?”
- This is the opposite of the earlier question: instead of being given a value and asked to find the area to the left of the value, now we are told the area to the left and asked to find the value
- With a table, we can perform this inverse search by finding the probability in the body of the table, then looking to the margins to find the percentile associated with it

## Calculating percentiles (cont'd)

- What is the 60th percentile of the normal curve?
- There is no “.600” in the table, but there is a “.599”, which corresponds to 0.25
- The real 60th percentile must lie between 0.25 and 0.26
- For this class, 0.25, 0.26, or anything in between is an acceptable answer
- Or we could obtain an exact answer from R:

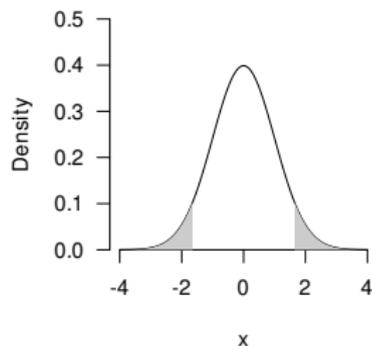
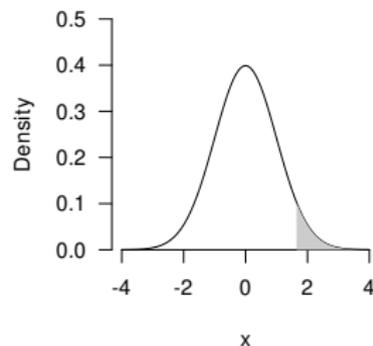
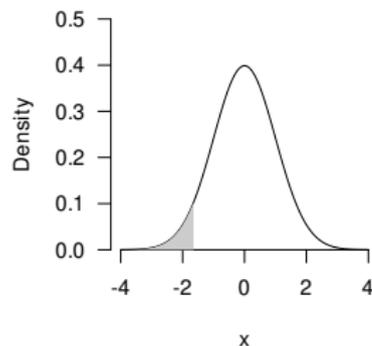
```
qnorm(0.6)  
# [1] 0.2533471
```

- How about the 10th percentile?

```
qnorm(0.1)  
# [1] -1.281552
```

# Calculating values such that a certain area lies within/outside them

Find the number  $x$  such that the area outside  $-x$  and  $x$  is equal to 10%



Our answer is therefore  $\pm 1.645$  (the 5th/95th percentile)

## Reconstructing a histogram

- In week 3, we said that the mean and standard deviation provide a two-number summary of a histogram
- We can now make this observation a little more concrete
- Anything we could have learned from the full data set, we will now determine by approximating the real distribution of the data by the normal distribution
- This approach is called the *normal approximation*

# NHANES adult women

- The data set we will work with on these examples is the NHANES sample of the heights of 2,649 adult women
- The mean height is 63.5 inches
- The standard deviation of height is 2.75 inches

## Procedure: Probabilities using the normal curve

The procedure for calculating probabilities with the normal approximation is as follows:

- (1) Draw a picture of the normal curve and shade in the appropriate probability
- (2) Convert to standard units: letting  $x$  denote a number in the original units and  $z$  a number in standard units,

$$z = \frac{x - \bar{x}}{SD}$$

where  $\bar{x}$  is the mean and  $SD$  is the standard deviation

- (3) Determine the area under the normal curve using a table or computer

## Estimating probabilities: Example # 1

- Suppose we want to estimate the percent of women who are under 5 feet tall
- 5 feet, or 60 inches is 1.27 standard deviations below the mean:  $(60 - 63.5)/2.75 = -1.27$
- Using the normal distribution, the probability of more than 1.27 standard deviations below the mean is  $P(x < -1.27) = 10.2\%$
- In the actual sample, 282 out of 2,649 women were under 5 feet tall, which comes out to 10.6%

## Estimating probabilities: Example # 2

- Another example: suppose we want to estimate the percent of women who are between 5'3 and 5'6 (63 and 66 inches)
- These heights are 0.18 standard deviations below the mean and 0.91 standard deviations above the mean, respectively
- Using the normal distribution, the probability of falling in this region is 39.0%
- In the actual data set, 1,029 out of 2,649 women were between 5'3 and 5'6: 38.8%

## Procedure: Percentiles using the normal curve

- We can also use the normal distribution to approximate percentiles
- The procedure for calculating percentiles with the normal approximation is as follows:
  - (1) Draw a picture of the normal curve and shade in the appropriate area under the curve
  - (2) Determine the percentiles of the normal curve corresponding to the shaded region using a table or computer
  - (3) Convert from standard units back to the original units:

$$x = \bar{x} + z(SD)$$

where, again,  $x$  is in original units,  $z$  is in standard units,  $\bar{x}$  is the mean, and  $SD$  is the standard deviation

## Approximating percentiles: Example

- Suppose instead that we wished to find the 75th percentile of these women's heights
- For the normal distribution, 0.67 is the 75th percentile
- The mean plus 0.67 standard deviations in height is 65.35 inches
- For the actual data, the 75th percentile is 65.39 inches

## The broad applicability of the normal approximation

- These examples are by no means special: the distribution of many random variables are very closely approximated by the normal distribution
- Indeed, this is why statisticians call it the “normal” distribution
- Other names for the normal distribution include the Gaussian distribution (Gauss contributed some important theory on its use) and the bell curve (after its shape)
- For variables with approximately normal distributions, the mean and standard deviation essentially tell us everything about the data – other summary statistics and graphics are redundant

## Caution

- Other variables, however, are not approximated by the normal distribution well, and give misleading or nonsensical results when you apply the normal approximation to them
- For example, the value 0 lies -1.23 standard deviations below the mean infant mortality rate for Europe
- The normal approximation therefore predicts a probability that 11% of the countries in Europe will have negative infant mortality rates

## Caution (cont'd)

- As another example, the normal distribution will always predict the median to lie 0 standard deviations above the mean
- *i.e.*, it will always predict that the median equals the mean
- As we have seen, however, the mean and median can differ greatly when distributions are skewed
- For example, according to the U.S. census bureau, the mean income in the United States is \$63,214, while the median income is \$44,225

# Summary

- The distribution of many random variables are very closely approximated by the normal distribution
- Know how to calculate the area under the normal curve
- Know how to determine percentiles of the normal curve
- Know how to approximate probabilities for real data using the normal approximation
- Know how to approximate quantiles for real data using the normal approximation