

## Maximum likelihood: Asymptotic normality

Patrick Breheny

October 22, 2025

# Intro

- Today, we continue with our goal of deriving the asymptotic properties of maximum likelihood estimators
- Previously, we established conditions under which the MLE was consistent:  $\|\hat{\theta} - \theta^*\| \xrightarrow{P} 0$
- Today, we will see that under those same conditions,  $\sqrt{n}(\hat{\theta} - \theta^*)$  converges in distribution to a multivariate normal
- After establishing this, we will consider how these results change if we remove the log-concavity assumption and allow for the possibility of multiple maxima

# Vector-valued functions

- The main idea behind the proof is to take a Taylor series expansion not of the likelihood, but rather the score
- However, this requires us to extend our earlier Taylor series results a bit
- Up until this point, we have always expanded scalar-valued functions, but the score function is vector-valued, which impacts the Lagrange form of the expansion

# Taylor series of vector-valued functions

- In particular, there is no longer a single point  $\bar{\theta}$  for which the Lagrange result holds, so we have to restate the assumptions somewhat and require some additional conditions
- **Theorem:** Suppose  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^k$  is twice differentiable on  $N_r(\mathbf{x}_0)$ , and that  $\nabla^2 f$  is bounded on  $N_r(\mathbf{x}_0)$ . Then for any  $\mathbf{x} \in N_r(\mathbf{x}_0)$ ,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \{\nabla \mathbf{f}(\mathbf{x}_0) + O(\|\mathbf{x} - \mathbf{x}_0\|)\}^\top (\mathbf{x} - \mathbf{x}_0)$$

## Application to the score

- Applying this expansion to the score vector, we obtain the following corollary, which will come up many times in this course
- **Corollary:** Suppose regularity conditions (A)-(C) from the previous lecture are met. Then for any consistent estimator  $\hat{\theta}$ , we have

$$\frac{1}{\sqrt{n}} \mathbf{u}(\hat{\theta}) = \frac{1}{\sqrt{n}} \mathbf{u}(\theta^*) - \{ \mathcal{J}(\theta^*) + o_p(1) \} \sqrt{n} (\hat{\theta} - \theta^*);$$

if  $\hat{\theta}$  is  $\sqrt{n}$ -consistent, then

$$\frac{1}{\sqrt{n}} \mathbf{u}(\hat{\theta}) = \frac{1}{\sqrt{n}} \mathbf{u}(\theta^*) - \mathcal{J}(\theta^*) \sqrt{n} (\hat{\theta} - \theta^*) + o_p(1)$$

## A corollary to the corollary

Similarly, for any two consistent estimators  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , we have

$$\frac{1}{\sqrt{n}}\mathbf{u}(\hat{\theta}_1) = \frac{1}{\sqrt{n}}\mathbf{u}(\hat{\theta}_2) - \{\mathcal{J}(\boldsymbol{\theta}^*) + o_p(1)\}\sqrt{n}(\hat{\theta}_1 - \hat{\theta}_2);$$

if both estimators are  $\sqrt{n}$ -consistent,

$$\frac{1}{\sqrt{n}}\mathbf{u}(\hat{\theta}_1) = \frac{1}{\sqrt{n}}\mathbf{u}(\hat{\theta}_2) - \mathcal{J}(\boldsymbol{\theta}^*)\sqrt{n}(\hat{\theta}_1 - \hat{\theta}_2) + o_p(1)$$

## Slutsky's extension

- Lastly, there is one more preliminary result that we need in order to prove asymptotic normality (I call this “Slutsky’s extension”, but I’m not aware of any established name for this result)
- **Theorem:** Suppose  $\mathbf{y}_n \xrightarrow{d} \mathbf{y}$ , where  $\mathbf{y}$  is a  $d \times 1$  random vector,  $\mathbf{A}_n \xrightarrow{P} \mathbf{A}$ , where  $\mathbf{A}$  is a positive definite matrix, and that  $\mathbf{y}_n = \mathbf{A}_n \mathbf{x}_n$ . Then

$$\mathbf{x}_n \xrightarrow{d} \mathbf{A}^{-1} \mathbf{y}$$

- Note that this would be trivial if  $\mathbf{A}_n$  were invertible for all  $n$ , but we don’t know that — only that converges to a positive definite matrix

# Asymptotic normality of the MLE

- Our main result for today is proving the following central limit theorem-like result for the MLE of any smooth log-concave model (which is pretty simple given all the earlier results)
- Theorem (Asymptotic normality of the MLE):** Suppose assumptions (A)-(D) from the previous lecture are met. Then the maximum likelihood estimator  $\hat{\theta}$  satisfies

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{d} N(\mathbf{0}, \mathcal{J}(\theta^*)^{-1}).$$

- Note that (A)-(D) therefore ensure not just consistency, but  $\sqrt{n}$ -consistency
- Note also another interpretation of the information: as information increases, the variance of the MLE  $\hat{\theta}$  decreases

# Non-standard problems

- Unlike the consistency proof, we do need differentiability requirements for asymptotic normality to hold
- For example, we remarked previously that for  $X_i \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$ , the MLE is consistent despite the likelihood not being continuous or differentiable at  $\theta^*$
- However, today's theorem does not hold for the uniform distribution:
  - Converges much faster:  $\hat{\theta} - \theta^*$  is  $O_p(1/n)$ , not  $O_p(1/\sqrt{n})$
  - $\mathcal{J}(\theta)$  is not even defined in the uniform case
  - Asymptotic distribution is not normal:

$$n(\theta^* - \hat{\theta}) \xrightarrow{d} \text{Exp}(1/\theta^*)$$

# Influence function

- Since the MLE is  $\sqrt{n}$ -consistent, we have

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) = \frac{1}{\sqrt{n}} \mathcal{J}^{-1}(\boldsymbol{\theta}^*) \mathbf{u}(\boldsymbol{\theta}^*) + o_p(1)$$

or in other words,

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^* + \frac{1}{n} \sum_i \mathcal{J}^{-1}(\boldsymbol{\theta}^*) \mathbf{u}_i(\boldsymbol{\theta}^*) + o_p(1/\sqrt{n})$$

- In statistics, this relationship is known as the *influence function* (formal definition on next slide); essentially, the derivative of the estimator with respect to the data

# A connection with nonparametric statistics

- Influence functions play an important role in semiparametric and nonparametric statistics
- Suppose we are interested in estimating some function  $T$  of a distribution  $F$ ; the influence function is defined as

$$\text{IF}(x) = \lim_{\epsilon \rightarrow 0} \left[ \frac{T\{(1 - \epsilon)F + \epsilon\delta_x\} - T(F)}{\epsilon} \right],$$

where  $\delta_x$  is a distribution with all of its mass at  $x$

- Then given some assumptions regarding the smoothness of  $T$ , the *von Mises expansion* essentially extends all of this Taylor series reasoning to the empirical CDF  $\hat{F}$ :

$$T(\hat{F}) = T(F) + \frac{1}{n} \sum \text{IF}_F(X_i) + o_p(1/\sqrt{n})$$

# Asymptotically linear estimators

- Such estimators are said to be *asymptotically linear*
  - All asymptotically linear estimators are therefore asymptotically normal by the CLT
  - And the influence function determines the asymptotic variance
- Influence functions therefore provide a unifying thread connecting parametric likelihood theory with semiparametric and nonparametric statistics
- The influence function is potentially much more complicated in such problems, but for the parametric MLE, it has a simple form:

$$\text{IF}(x) = \mathcal{J}^{-1}(\boldsymbol{\theta}^*) \mathbf{u}(\boldsymbol{\theta}^* | x)$$

# Multiple roots

- Finally, let's consider what happens if we drop assumption (D), that our likelihood is log-concave
- In this case, there are potentially many solutions to the likelihood equations

$$\mathbf{u}(\boldsymbol{\theta}) = \mathbf{0},$$

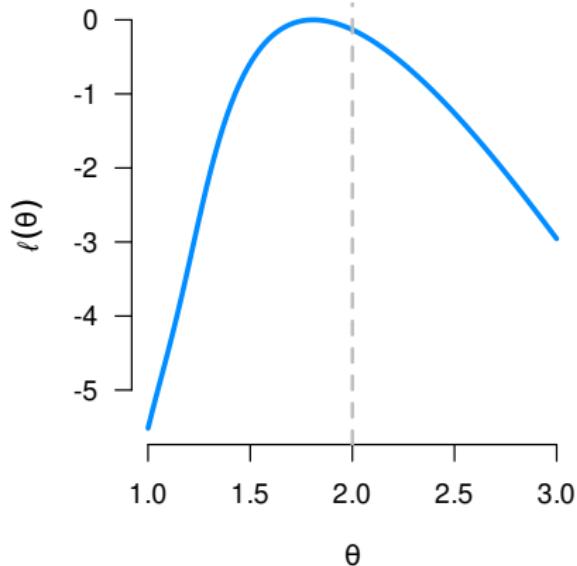
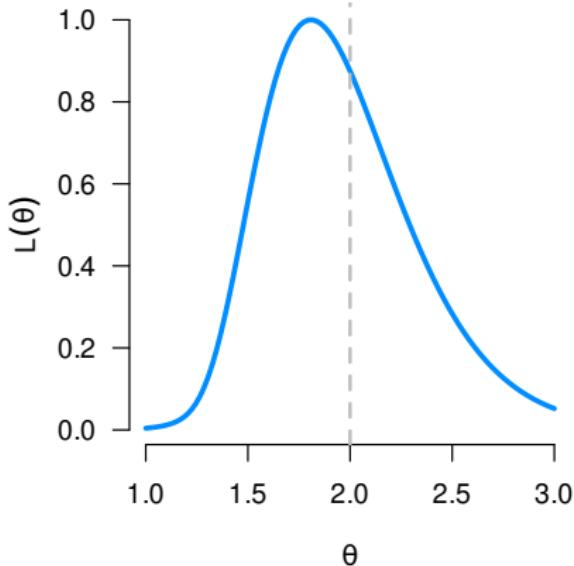
- even if the MLE is unique
- Furthermore, as our counterexample at the beginning of the last lecture shows, if the likelihood has multiple modes there is no guarantee that the MLE is even consistent

## Local log-concavity

- However, as you probably noticed, when proving consistency we only used assumption (D) at the very last step
- If we remove assumption (D), every step of the proof remains, except for the fact that at the end, all we can say is that there is a local maximum (i.e., a solution to the likelihood equations, not *the* solution to the likelihood equations) inside  $\Theta^*$  that is consistent and asymptotically normal
- In other words, the likelihood isn't log-concave everywhere, but if the other conditions are met, and in particular if  $\mathcal{J}(\theta^*)$  is positive definite, then there is a neighborhood  $\Theta^*$  inside of which the likelihood is log-concave, and our theorems hold in a local sense

## Revisiting our inconsistent MLE

The MLE isn't consistent but there is local solution which is:



## Restating our earlier theorems

- With this in mind, we can offer more general restatements of our earlier theorems
- Theorem (Consistency of the MLE):** Suppose assumptions (A)-(C) are met. Then with probability tending to 1, there exists a consistent sequence of solutions  $\hat{\theta}_n$  to the likelihood equations:

$$\hat{\theta}_n \xrightarrow{P} \theta^*.$$

- Theorem (Asymptotic normality of the MLE):** Suppose assumptions (A)-(C) from the previous lecture are met. Then with probability tending to 1, there exists a consistent sequence of solutions  $\hat{\theta}_n$  to the likelihood equations satisfying

$$\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(\mathbf{0}, \mathcal{J}(\theta^*)^{-1}).$$

# Useful?

- Now, is this a useful generalization?
- Not necessarily — in theory, you can just choose the correct root, but that's an extremely strong assumption
  - Whatever algorithm we're using to maximize the likelihood is probably only going to return a single solution — we have no guarantees about its properties
  - Even if we were able to find all solutions of the likelihood equations, we have no way of knowing which one to choose

## Useful? (cont'd)

- But also ... maybe?
- Suppose we have an estimator  $\tilde{\theta}$ , not the MLE, that we knew to be consistent
- We could, for example, pick the solution to the likelihood equations closest to  $\tilde{\theta}$
- More ambitiously, we could take a Taylor series expansion of the likelihood equations about the point  $\tilde{\theta}$ , then estimate  $\theta$  via:

$$\hat{\theta} = \tilde{\theta} + \mathcal{I}_n(\tilde{\theta})^{-1} \mathbf{u}(\tilde{\theta})$$

- You can iterate this process if desired, repeating the above calculation until convergence (this is Newton's method), or just stop after one application (the "one-step estimator")

# One-step estimator theorem

- We'll skip the proof of this, but if  $\tilde{\theta}$  is not only consistent but  $\sqrt{n}$ -consistent, then our results hold not just for some mysterious, unknown root of the likelihood equations, but for the unique root defined on the previous slide
- **Theorem:** Suppose conditions (A)-(C) from the previous lecture are met, and that  $\tilde{\theta}$  is a  $\sqrt{n}$ -consistent estimator of  $\theta$ . Define  $\hat{\theta}_n = \tilde{\theta}_n + \mathcal{I}_n^{-1}(\tilde{\theta}_n)\mathbf{u}(\tilde{\theta}_n)$ . Then

$$\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(\mathbf{0}, \mathcal{J}(\theta^*)^{-1}).$$

- One can also use  $\mathcal{J}(\tilde{\theta})$  to construct  $\hat{\theta}$  and the theorem still holds

## Cauchy example

- For example, suppose  $X_i \stackrel{\text{iid}}{\sim} \text{Cauchy}(\theta)$ ; as we have already seen, this likelihood has multiple local maxima and it is unclear whether any given solution to the likelihood equations is consistent and asymptotically normal
- However, it can be shown that the sample median,  $\tilde{\theta}$ , is not the MLE but is a  $\sqrt{n}$ -consistent estimator of  $\theta$
- Thus, the procedure on the previous slide can be used to obtain the likelihood root with known consistency and asymptotic normality properties

## A word of caution

- The Cauchy distribution is a nice success story of maximum likelihood in the presence of multiple roots, but is arguably more of the exception than the rule
- In practice, it's inherently risky to go around constructing inference based on maximum likelihood in the presence of a likelihood with multiple maxima
- In other words, don't misinterpret the theory to mean that *any* root of the likelihood equations will be asymptotically normal