Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 1
Due: Tuesday, September 1

1. *Exchange paradox.* Consider the following seemingly paradoxical situation. There are two identical envelopes, each containing money. One contains twice as much as the other. You may pick one envelope and keep the money it contains. You randomly choose an envelope, open it, and learn that it contains \$20. You are then given the opportunity to switch envelopes. Should you? You reason as follows: "The other envelope contains either \$10 or \$40 dollars, with equal probability. Therefore, if I switch envelopes, my expected value is

$$\tfrac{1}{2} \cdot 10 + \tfrac{1}{2} \cdot 40 = 25,$$

   which is larger than the \$20 dollars I can expect if I keep the original envelope." So you switch.

   This seems unreasonable, however, since the argument would be exactly the same regardless of the value of your envelope: if you learn that your envelope contains x dollars, the reasoning above concludes that the other envelope contains $5x/4$ dollars. If we should always switch envelopes, regardless of how much money our envelope contains, it would seem that something has gone wrong.

   Explain what is wrong with the above argument. There are many possible solutions, although in keeping with the spirit of the course, I would encourage you to work the concept of likelihood into your answer.

2. *Combining likelihoods, part 1.* Suppose we have two independent samples drawn from $N(\theta, 1)$. All we know about the first sample is that $n_1 = 5$ and the maximum value of these five observations is 3.5. For the second sample, we know that $n_2 = 3$ and $\bar{x}_2 = 4$.

   (a) Derive the combined likelihood.

   (b) Plot the following three likelihood curves (within a single graph):

   - The likelihood from sample 1
   - The likelihood from sample 2
   - The combined likelihood

   All three likelihoods should be scaled such that their maximum value is 1. Comment on how the combined likelihood compares to the likelihoods from samples 1 and 2.

3. *Combining likelihoods, part 2.* In the previous exercise, we combined density and probability terms …maybe this causes problems? Discretize the combined likelihood from the previous exercise: instead of assuming that we observe the summary statistic exactly, assume that the summary statistic is observed to fall with the window $x - \epsilon < X < x + \epsilon$ so that the likelihood is based only on probabilities, not densities.

   Plot the following three likelihood curves within a single graph:

   - The original combined likelihood (with density terms)
   - The "discretized" likelihood with $\epsilon = 0.1$

- The “discretized” likelihood with $\epsilon = 0.5$

Provide two separate plots: one in which the likelihoods are on their “raw” scale and the other in which they are normalized such that their maximum value is 1.

In addition to the plots themselves:

- Discuss the pros and cons of each version of the plot (raw and normalized versions) in terms of what they communicate.
- Discuss whether combining density and probability is an issue for likelihood-based inference.