

Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 6
Due: Tuesday, October 6

1. *Exponential Taylor series.*

- (a) Show that for any x ,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note: one can partially prove this using the Poisson distribution, but this proof would only work for $x > 0$.

- (b) Starting with the result in (a), derive the infinite series for b^x , where $b > 0$.
(c) Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$. What is the second-order Taylor series for $f(\mathbf{x}) = \exp(\mathbf{a}^\top \mathbf{x})$ about $\mathbf{x} = \mathbf{0}$? Give both the o -notation and Lagrange forms.
(d) Suppose $\mathbf{a} = [2 \quad -1]^\top$ and $\mathbf{x} = [1 \quad 1]^\top$. Find the point $\bar{\mathbf{x}}$ on the line segment connecting $\mathbf{x}$ and $\mathbf{0}$ that satisfies the Lagrange form of Taylor's theorem.
2. *Slutsky constant.* In Slutsky's theorem, we require that one of the two random variables converges in probability to a constant. What happens if both $\mathbf{x}_n$ and $\mathbf{y}_n$ converge in distribution? Specifically, suppose $\mathbf{x}_n \xrightarrow{d} \mathbf{x}$ and $\mathbf{y}_n \xrightarrow{d} \mathbf{y}$; is it true that

$$\begin{bmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}?$$

For each of the following scenarios, either prove that this is true or prove that it is not by providing a counterexample.

- (a) Without any additional restrictions.
(b) With the added restriction that $\mathbf{x}_n \perp \mathbf{y}_n$ and $\mathbf{x} \perp \mathbf{y}$.
3. *Exponential dispersion.* For the exponential dispersion family,

$$\begin{aligned} \mathbb{E}(\mathbf{s}) &= \nabla \psi(\boldsymbol{\theta}) \\ \mathbb{V}(\mathbf{s}) &= \phi \nabla^2 \psi(\boldsymbol{\theta}). \end{aligned}$$

In this problem, we will derive these results using two different approaches. Note that the first (using properties of the score) is obviously much easier if you've already derived those properties, but it relies on doing quite a bit of work to prove those theorems in the first place.

Both approaches require the ability to change the order of derivation and integration: this is always justified for exponential families (you do not need to show it). Note that in part (a), you are starting with the derivative inside the integral and moving it out, while in part (b), you are starting with the derivative outside the integral and moving it in.

Just do this for a single observation. Nothing changes when additional observations are involved.

- (a) Derive the above results using the expectation and variance of the score (very simple, one or two lines).
- (b) Derive the above results starting from the fact that

$$\exp\{\psi(\boldsymbol{\theta})/\phi\} = \int \exp\{\mathbf{s}^\top \boldsymbol{\theta}/\phi\} dP(x|\phi),$$

then taking first and second derivatives (this approach is longer).