

Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 8
Due: Tuesday, October 27

1. *Regularity conditions.* Suppose $X_1, \dots, X_n$ are an iid sample with density

$$p(x|\theta) = \theta x^{\theta-1} 1\{x \in (0, 1)\}, \quad \Theta = (0, \infty).$$

- (a) Check whether this model satisfies regularity conditions (A)–(D) as defined in the notes. For the purposes of the problem, I would like you to explicitly check each condition (as opposed to just using a general result about exponential families).
- You may assume condition C(i) holds; verifying the exchange of differentiation and integration is a bit tedious and we'll discuss it in class instead.
 - For (B) and (C), explicitly specify the open neighborhood Θ^* use the same Θ^* throughout.
- (b) Find the MLE and its limiting distribution (i.e., a convergence in distribution statement involving some function of the MLE; the MLE itself just converges to θ^* in probability).
2. *Combination of Poisson variables.* Suppose in one month, a public health department records X cases of a new disease. At the end of the month, it is discovered that this disease is really two different diseases with similar symptoms. In month two, data recording practices are changed and the number of cases of each disease, Y_1 and Y_2 , is recorded separately. For the purposes of the problem, assume that the number of cases of disease i in a month follows a Poisson distribution with rate $n\lambda_i$, where the population size n is known and constant from one month to the next, that cases of the two diseases arise independently, and that the number of cases in one month is independent of the number of cases in the next month.
- (a) Find the MLE of λ_1 and λ_2 .
- (b) Suppose n is large enough that we feel comfortable applying the central limit theorem to (X, Y_1, Y_2) . In other words, you may apply the approximation given by

$$\sqrt{n}(Y_1/n - \lambda_1) \xrightarrow{d} N(0, \lambda_1);$$

and so on for X and Y_2 . What is the approximate variance of $\hat{\lambda}_1$ as a function of n , λ_1 , and λ_2 ?

3. *Slutsky's extension.* Suppose $\mathbf{y}_n \xrightarrow{d} \mathbf{y}$, where $\mathbf{y}$ is a $d \times 1$ random vector, $\mathbf{A}_n \xrightarrow{P} \mathbf{A}$, where $\mathbf{A}$ is a positive definite matrix, and that $\mathbf{y}_n = \mathbf{A}_n \mathbf{x}_n$. Note: This problem would be trivial if we knew that $\mathbf{A}_n$ were positive definite; the point of this problem is that you do *not* know this about $\mathbf{A}_n$, only that its limit $\mathbf{A}$ is positive definite. In other words, if at any point you write $\mathbf{A}_n^{-1}$ in your solution, then it is incorrect.

- (a) Find a lower bound $L > 0$ such that

$$\mathbb{P} \left\{ \frac{\mathbf{x}_n^\top \mathbf{A}_n^\top \mathbf{A}_n \mathbf{x}_n}{\|\mathbf{x}_n\|^2} > L \right\} \rightarrow 1.$$

Hint: If $\mathbf{A}$ is positive definite, $\mathbf{A}^\top \mathbf{A}$ is positive definite.

- (b) Prove that $\mathbf{x}_n$ is bounded in probability. Hint: Use the result from (a).
- (c) Prove that $\mathbf{x}_n \xrightarrow{d} \mathbf{A}^{-1} \mathbf{y}$. Hint: Use the result from (b).